TreeKs: a Functor to Make Abstract Numerical Domains Scalable

Research Internship, advised by Antoine Miné
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Motivation and context

Abstract interpretation is a formal theory of sound approximation of semantics, mainly used in static analyzer, such as:

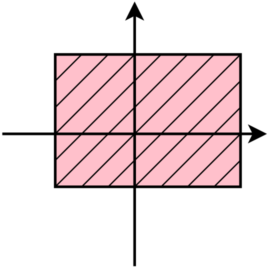
- Clousot: static verification of Code Contracts
- Astrée: proof of absence of runtime errors on embedded softwares

Abstract numerical domains:

- a set $\mathcal{D}_\mathcal{V}$ of computer-representable abstract values
- effective algorithms to compute sound abstractions of the operations: intersection $\sqcap^{\mathcal{D}_\mathcal{V}}$, union $\sqcup^{\mathcal{D}_\mathcal{V}}$, projection $\exists^{\mathcal{D}_\mathcal{V}}$, ...

Numerical abstract domains: examples

Intervals [Cousot Cousot 76]

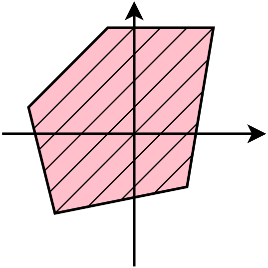


$\bigwedge_i a_i \leq X_i \leq b_i$

Non-relational

Linear cost

Polyhedra [Cousot Halbwachs 78]



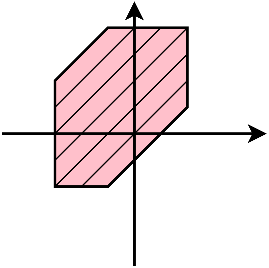
$\bigwedge_j \sum_i a_{ij} X_i \leq b_j$

Relational and very precise

Worst-case exponential cost

Weakly relational numerical abstract domains

Zones [Miné 01]



$\bigwedge_{ij} X_i - X_j \leq c_{ij}$

Weakly relational
Cubic cost

Octagons [Miné 01]
$\bigwedge_{ij} \pm X_i \pm X_j \leq c_{ij}$
Cubic cost

Logahedra [Howe King 09]
$\bigwedge_{ij} \pm 2^{a_i} X_i \pm 2^{b_j} X_j \leq c_{ij}$
Cubic cost

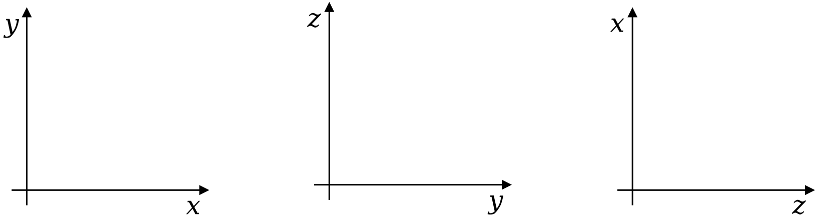
TVPI [Simon King Howe 02]
$\bigwedge_{ij} a_i X_i + b_j X_j \leq c_{ij}$
Quasi-cubic cost

Octahedra [Clarisó Cortadella 07]

$\bigwedge \sum_i \pm X_i \leq c$
Worst-case exponential cost

Closure operation: example

Domain of zones $(\bigwedge_{ij} X_i - X_j \leq b_{ij})$
$\mathcal{V} = \{x, y, z\}$



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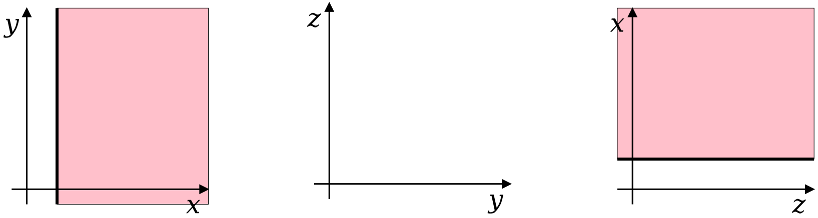

$-x \leq -1$

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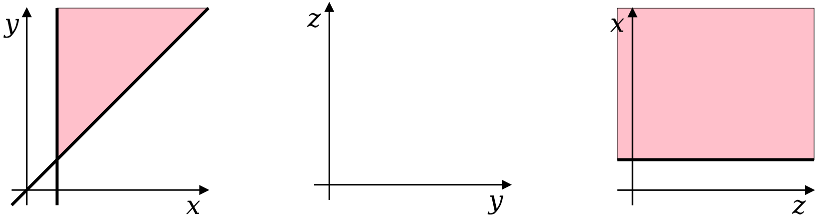

$-x \leq -1$
$x - y \leq 0$

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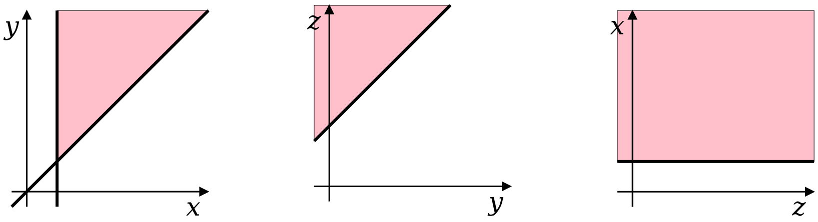

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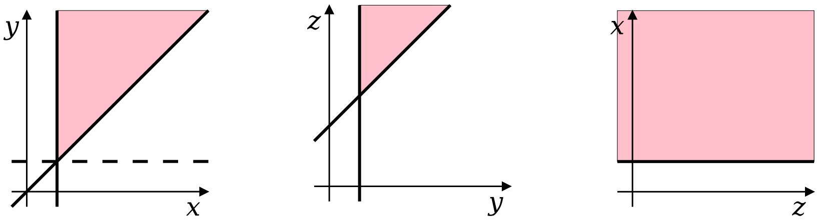

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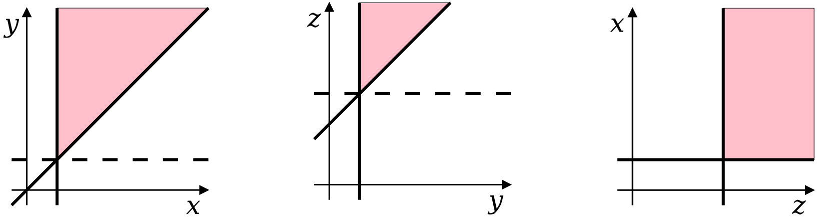

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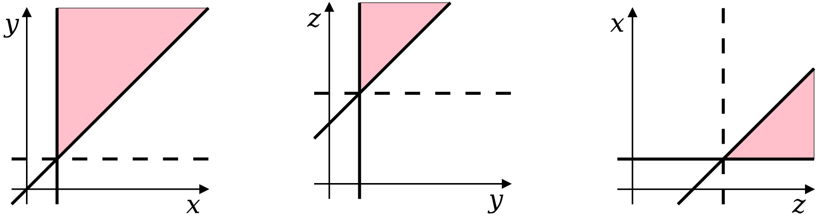

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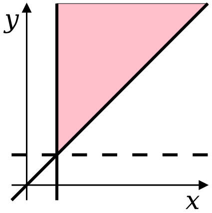
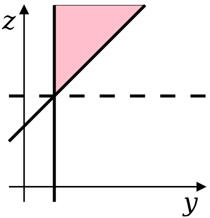
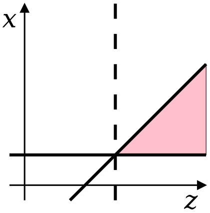




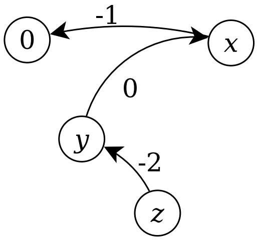


$-x \leq -1$
$x - y \leq 0$
$y - z \leq -2$

$-y \leq -1$
$-z \leq -3$
$x - z \leq -2$
Done!

Domain of zones: representation

We represent a set of difference constraints between two variables ($X_i - X_j \leq \mathbf{m}_{ji}$) by a potential graph or by a DBM (*Difference Bound Matrix*).



	0	x	y	z
0	0	$+\infty$	$+\infty$	$+\infty$
x	-1	0	$+\infty$	$+\infty$
y	$+\infty$	0	0	$+\infty$
z	$+\infty$	$+\infty$	-2	0

$$0 - x \leq -1$$
$$x - y \leq 0$$
$$y - z \leq -2$$

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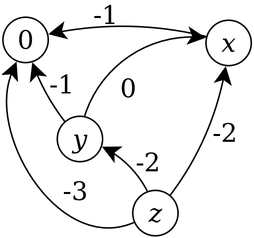


	0	x	y	z
0	0	$+\infty$	$+\infty$	$+\infty$
x	-1	0	$+\infty$	$+\infty$
y	-1	0	0	$+\infty$
z	-3	-2	-2	0

$$\begin{aligned} 0 - x &\leq -1 \\ x - y &\leq 0 \\ y - z &\leq -2 \end{aligned}$$

$$\begin{aligned} 0 - y &\leq -1 \\ 0 - z &\leq -3 \\ x - z &\leq -2 \end{aligned}$$

Domain of zones: closure and other operators

The closure is a shortest-path closure.

After closure, operators are point-wise.

Join (best approximation of union):

$$(\mathbf{m} \sqcup \mathbf{n})_{ij} = \max(\mathbf{m}_{ij}, \mathbf{n}_{ij})$$

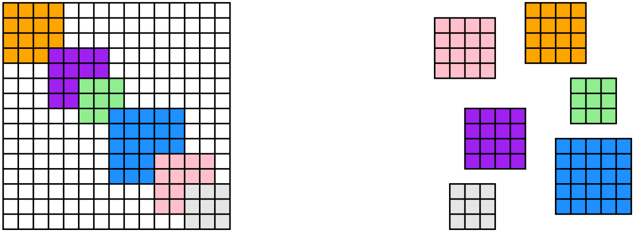
Forget operator (projection):

$$(\exists_{X_k} \mathbf{m})_{ij} = \begin{cases} \mathbf{m}_{ij} & \text{if } i \neq k \text{ and } j \neq k \\ 0 & \text{if } i = j = k \\ +\infty & \text{otherwise} \end{cases}$$

How to scale: packing

Principle:

- split variables into packs
- use a DBM per pack



Cost: linear for bounded-size packs
Information loss: no communication between packs!

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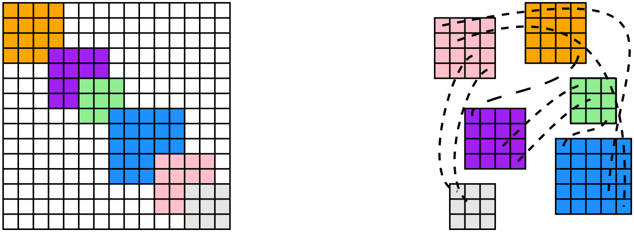


Cost: linear for bounded-size packs

Information loss: no communication between packs!

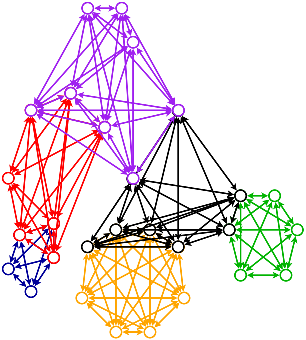
Solution: intervals constraints sharing

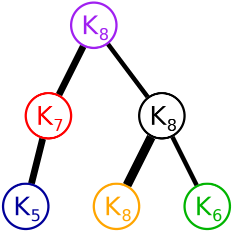
Not good enough!

TreeKs: a certain subgraph

Shape:

- a tree of complete graphs (packs)
- sharing borders





packs tree

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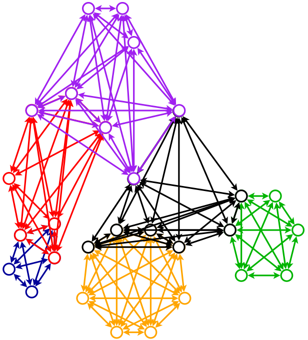

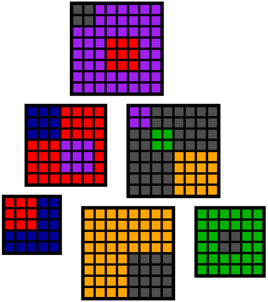

Abstract value: tuple of DBMs

Closure algorithm

Closure algorithm in TreeKs $O(mp^3)$

for each *pack from the leaves to the root*
- Apply closure on this pack in the domain of zones
- Pass the new constraints to his father

for each *pack from the root to the leaves*
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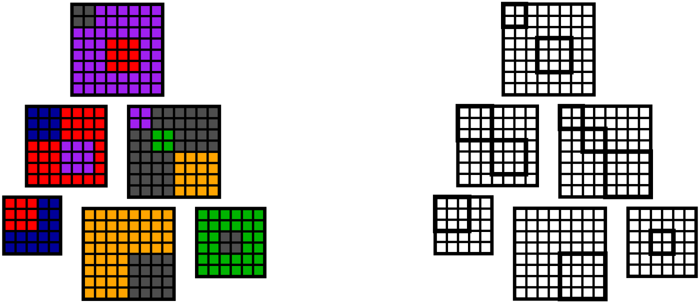

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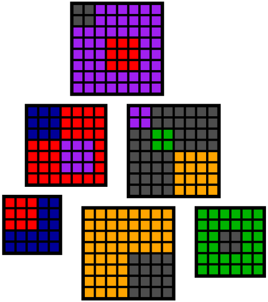

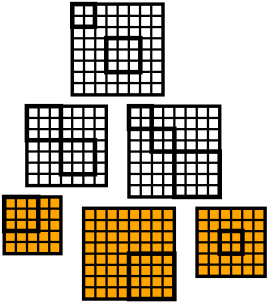

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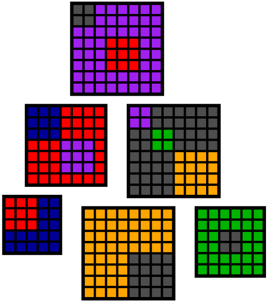

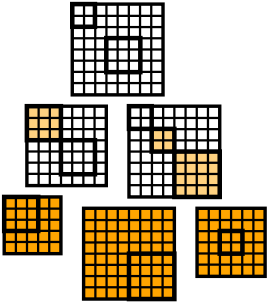

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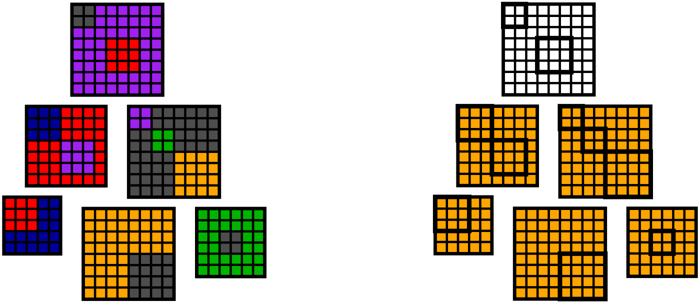

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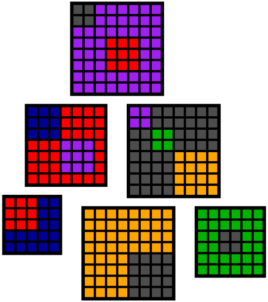

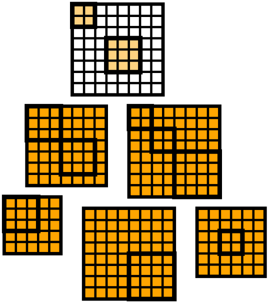

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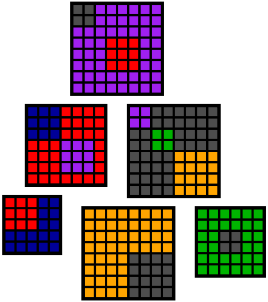

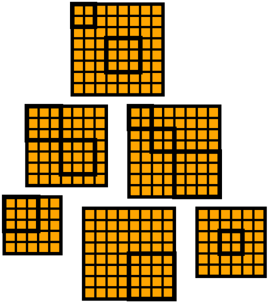

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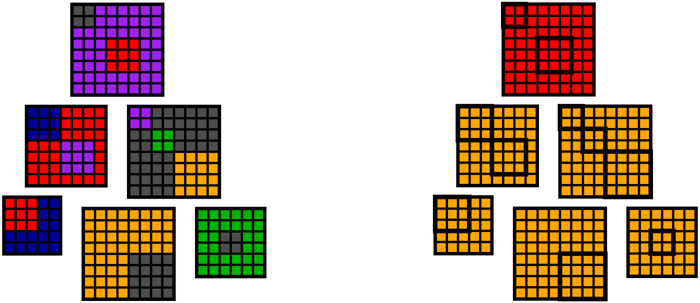

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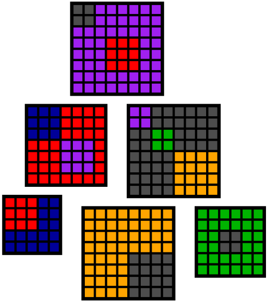

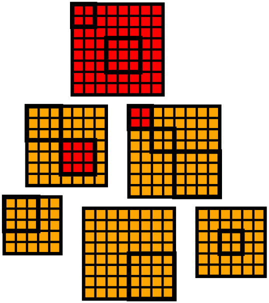

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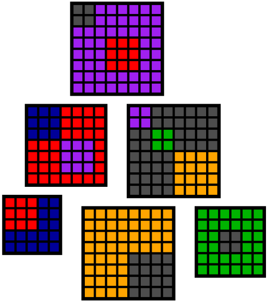

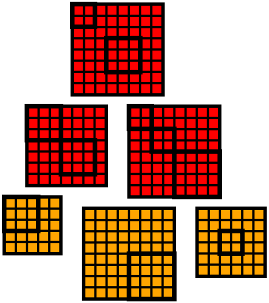

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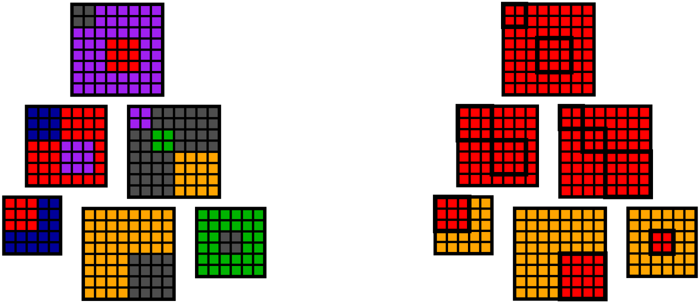

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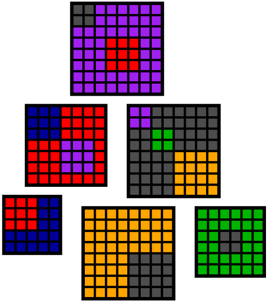

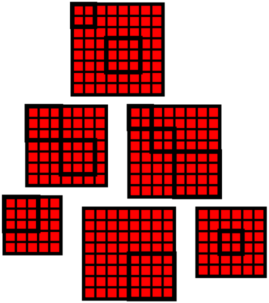

Conclusion

- can be applied to many numerical abstract domains (zones, octagons, logahedra, TVPI, octahedra, polyhedra, ...)
- linear cost when pack size is bounded

Future work:

- implementation
- development of packs generation strategies
- application to other domains